

**ABSTRACT**

Time series modeling and forecasting has fundamental importance to various practical domains. Thus a lot of active research works is going on in this subject during several years. Many important models have been proposed in literature for improving the accuracy and efficiency of time series modeling and forecasting. The aim of this paper is to present a concise description of some popular time series modeling and forecasting using stochastic models with their salient features.

**KEYWORDS:** Forecasting, Stochastic Model

**INTRODUCTION**

In general models for time series data can have many forms and represent different stochastic processes. There are two widely used linear time series models in literature, viz. Autoregressive (AR) and Moving Average (MA) models. Combining these two, the Autoregressive Moving Average (ARMA) and Autoregressive Integrated Moving Average (ARIMA) models have been proposed in literature. The Autoregressive Fractionally Integrated Moving Average (ARFIMA) model generalizes ARMA and ARIMA models. For seasonal time series forecasting, a variation of ARIMA, viz. the Seasonal Autoregressive Integrated Moving Average (SARIMA) model is used. ARIMA model and its different variations are based on the famous Box-Jenkins principle and so these are also broadly known as the Box-Jenkins models.

Linear models have drawn much attention due to their relative simplicity in understanding and implementation. However many practical time series show non-linear patterns. For example, as mentioned by R. Parrelli, non-linear models are appropriate for predicting volatility changes in economic and financial time series. Considering these facts, various non-linear models have been suggested in literature. Some of them are the famous Autoregressive Conditional Heteroskedasticity (ARCH) model and its variations like Generalized ARCH (GARCH), Exponential Generalized ARCH (EGARCH) etc., the Threshold Autoregressive (TAR) model, the Non-linear Autoregressive (NAR) model, the Non-linear Moving Average (NMA) model, etc. In the present paper we discuss about the important linear and non-linear stochastic time series models with their different properties.

**THE AUTOREGRESSIVE MOVING AVERAGE (ARMA) MODELS**

An ARMA( $p, q$ ) model is a combination of AR( $p$ ) and MA( $q$ ) models and is suitable for univariate time series modeling. In an AR( $p$ ) model the future value of a variable is assumed to be a linear combination of  $p$  past observations and a random error together with a constant term. Mathematically the AR( $p$ ) model can be expressed as:

$$y_t = c + \sum_{i=1}^p \phi_i y_{t-i} + \varepsilon_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t \quad (1)$$

Here  $y_t$  and  $\varepsilon_t$  are respectively the actual value and random error (or random shock) at time period  $t$ ,  $\phi_i$  ( $i = 1, 2, \dots, p$ ) are model parameters and  $c$  is a constant. The integer constant  $p$  is known as the order of the model. Sometimes the constant term is omitted for simplicity. Usually For estimating parameters of an AR process using

the given time series, the Yule-Walker equations are used. Just as an AR( $p$ ) model regress against past values of the series, an MA( $q$ ) model uses past errors as the explanatory variables. The MA( $q$ ) model is given by

$$y_t = \mu + \sum_{j=1}^q \theta_j \varepsilon_{t-j} + \varepsilon_t = \mu + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q} + \varepsilon_t \quad (2)$$

Here  $\mu$  is the mean of the series,  $\theta_j$  ( $j = 1, 2, \dots, q$ ) are the model parameters and  $q$  is order of the model. The random shocks are assumed to be a white noise process, i.e. a sequence of independent and identically distributed (i.i.d) random variables with zero mean and a constant variance  $\sigma^2$ . Generally, the random shocks are assumed to follow the typical normal distribution. Thus conceptually a moving average model is a linear regression of the current observation of the time series against the random shocks of one or more prior observations. Fitting an MA model to a time series is more complicated than fitting an AR model because in the former one the random error terms are not fore-seeable.

Autoregressive (AR) and moving average (MA) models can be effectively combined together to form a general and useful class of time series models, known as the ARMA models. Mathematically an ARMA( $p, q$ ) model is represented as

$$y_t = c + \varepsilon_t + \sum_{i=1}^p \phi_i y_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} \quad (3)$$

Here the model orders  $p, q$  refer to  $p$  autoregressive and  $q$  moving average terms.

Usually ARMA models are manipulated using the lag operator notation. The lag or backshift operator is defined as  $Ly_t = y_{t-1}$ . Polynomials of lag operator or lag polynomials are used to represent ARMA models as follows

$$\text{AR}(p) \text{ model: } \varepsilon_t = \phi(L) y_t$$

It is shown that an important property of AR( $p$ ) process is invertibility, i.e. an AR( $p$ ) process can always be written in terms of an MA( $\infty$ ) process. Whereas for an MA( $q$ ) process to be invertible, all the roots of the equation  $\theta(L) = 0$  must lie outside the unit circle. This condition is known as the Invertibility Condition for an MA process.

### STATIONARITY ANALYSIS

When an AR( $p$ ) process is represented as  $\varepsilon_t = \phi(L) y_t$ , then  $\phi(L) = 0$  is known as the Characteristic equation for the process. It is proved by Box and Jenkins that a necessary and sufficient condition for the AR( $p$ ) process to be stationary is that all the roots of the characteristic equation must fall outside the unit circle. Hipel and McLeod mentioned another simple algorithm (by Schur and Pagano) for determining stationarity of an AR process. For example as shown in the AR(1) model  $y_t = c + \phi_1 y_{t-1} + \varepsilon_t$  is stationary when  $\phi_1 < 1$  with constant mean and variance.

An MA( $q$ ) process is always stationary, irrespective of the values the MA parameters. The conditions regarding stationarity and invertibility of AR and MA processes also hold for an ARMA process. An ARMA( $p, q$ ) process is stationary if all the roots of the characteristic

equation  $\phi(L) = 0$  lie outside the unit circle. Similarly, if all the roots of the lag equation

$\theta(L) = 0$  lie outside the unit circle, then the ARMA( $p, q$ ) process is invertible and can be expressed as a pure AR process.

### AUTOREGRESSIVE INTEGRATED MOVING AVERAGE (ARIMA) MODELS

The ARMA models, described above can only be used for stationary time series data. However in practice many time series such as those related to socio-economic and business show non-stationary behavior. Time series, which contain trend and seasonal patterns, are also non-stationary in nature. Thus from application view point ARMA models are inadequate to properly describe non-stationary time series, which are frequently encountered in practice. For this reason the ARIMA model is proposed, which is a generalization of an ARMA model to include the case of non-stationarity as well.

In ARIMA models a non-stationary time series is made stationary by applying finite differencing of the data points. The mathematical formulation of the ARIMA( $p, d, q$ ) model using lag polynomials is given below

$$\phi(L)(1-L)^d y_t = \theta(L)\varepsilon_t, \text{ i.e.} \quad (4)$$

$$1 - \sum_{i=1}^p \phi_i L^i (1-L)^d y_t = 1 + \sum_{j=1}^q \theta_j L^j \varepsilon_t$$

- Here,  $p$ ,  $d$  and  $q$  are integers greater than or equal to zero and refer to the order of the autoregressive, integrated, and moving average parts of the model respectively.
- The integer  $d$  controls the level of differencing. Generally  $d=1$  is enough in most cases. When  $d=0$ , then it reduces to an ARMA( $p, q$ ) model.
- An ARIMA( $p, 0, 0$ ) is nothing but the AR( $p$ ) model and ARIMA( $0, 0, q$ ) is the MA( $q$ ) model.
- ARIMA( $0, 1, 0$ ), i.e.  $y_t = y_{t-1} + \varepsilon_t$  is a special one and known as the *Random Walk* model.

It is widely used for non-stationary data, like economic and stock price series.

A useful generalization of ARIMA models is the Autoregressive Fractionally Integrated Moving Average (ARFIMA) model, which allows non-integer values of the differencing parameter  $d$ . ARFIMA has useful application in modeling time series with long memory. In this model the expansion of the term  $(1-L)^d$  is to be done by using the general binomial theorem. Various contributions have been made by researchers towards the estimation of the general ARFIMA parameters.

### SEASONAL AUTOREGRESSIVE INTEGRATED MOVING AVERAGE (SARIMA) MODELS

The ARIMA model (4) is for non-seasonal non-stationary data. Box and Jenkins have generalized this model to deal with seasonality. Their proposed model is known as the Seasonal ARIMA (SARIMA) model. In this model seasonal differencing of appropriate order is used to remove non-stationarity from the series. A first order seasonal difference is the difference between an observation and the corresponding observation from the previous year and is calculated as  $z_t = y_t - y_{t-s}$ . For monthly time series  $s = 12$  and for quarterly time series

$s = 4$ . This model is generally termed as the SARIMA( $p, d, q$ )  $\times$  ( $P, D, Q$ ) $^s$  model.

The mathematical formulation of a SARIMA( $p, d, q$ )  $\times$  ( $P, D, Q$ ) $^s$  model in terms of lag polynomials is given below

$$\Phi_P(L^s)\phi_p(L)(1-L)^d(1-L^s)^D y_t = \Theta_Q(L^s)\theta_q(L)\varepsilon_t, \quad (5)$$

$$\text{i.e. } \Phi_P(L^s)\phi_p(L)z_t = \Theta_Q(L^s)\theta_q(L)\varepsilon_t.$$

Here  $z_t$  is the seasonally differenced series.

### SOME NONLINEAR TIME SERIES MODELS

So far we have discussed about linear time series models. As mentioned earlier, nonlinear models should also be considered for better time series analysis and forecasting. Campbell, Lo and McKinley (1997) made important contributions towards this direction. According to them almost all non-linear time series can be divided into two branches: one includes models non-linear in mean and other includes models non-linear in variance (heteroskedastic). As an illustrative example, here we present two nonlinear time series models from

- *Nonlinear Moving Average (NMA) Model*:  $y_t = \varepsilon_t + \alpha \varepsilon_{t-1}$ . This model is non-linear in mean but not in variance.
- *Eagle's (1982) ARCH Model*:  $y_t = \varepsilon_t + \alpha \varepsilon_t^2$ . This model is heteroskedastic, i.e. non-linear in variance, but linear in mean. This model has several other variations, like GARCH, EGARCH etc.

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